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# Nonsmooth Invex Functions and Sufficient Optimality Conditions

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In this paper the various definitions of nonsmooth invex functions are gathered in a general scheme by means of the concept of $K$-directional derivative. Characterizations of nonsmooth $K$-invexity are derived as well as results concerning constrained optimization without any assumption of convexity of the $K$-directional derivatives. © 2001 Academic Press

## 1. INTRODUCTION AND NOTATIONS

In [15], Hanson presented a weakened concept of convexity for differentiable functions:

- a differentiable function $f$: $X \to \mathbb{R}$ is said to be *invex*, if there exists a function $\eta$: $X \times X \to \mathbb{R}^n$ such that

$$f(x_2) - f(x_1) \geq \langle \nabla f(x_1), \eta(x_1, x_2) \rangle, \qquad \forall x_1, x_2 \in X.$$

The name invex descends from a contraction of "invariant convex" and it was proposed by Craven [6]. In [7], Craven and Glover showed that the class of invex functions is equivalent to the class of functions whose stationary points are global minima. In [18], Kaul and Kaur considered the following generalizations:

- $f$ is said to be *pseudoinvex* if there exists a function $\eta$: $X \times X \to \mathbb{R}^n$ such that

$$\langle \nabla f(x_1), \eta(x_1, x_2) \rangle \geq 0 \Rightarrow f(x_2) \geq f(x_1), \qquad \forall x_1, x_2 \in X;$$



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- $f$ is said to be *quasi-invex* if there exists a function $\eta$: $X \times X \to \mathbb{R}^n$ such that

$$f(x_2) \leq f(x_1) \Rightarrow \langle \nabla f(x_1), \eta(x_1, x_2) \rangle \leq 0, \qquad \forall x_1, x_2 \in X.$$

By means of these concepts they established sufficient optimality conditions for a nonlinear programming problem with inequality constraints. In [1, 11] relations among convex and invex functions and their generalizations were studied.

Recently Reiland [22] extended the concept of invexity to the class of locally Lipschitz functions using the generalized gradient of Clarke. An analogous extension was made by Jeyakumar [17] by means of the notion of approximate quasidifferentiability for nonsmooth functions. Different definitions of nonsmooth invexity were introduced by Ye [26] and Giorgi and Guerraggio [12] who studied the relations among all these classes of functions.

In this paper we propose a unifying definition of invexity for nonsmooth functions exploiting the concept of *local cone approximation* introduced in [9]. Moreover, via such an approach, we give sufficient optimality conditions for inequality constrained extremum problems without requiring the convexity of the directional derivatives.

In the sequel $X \subseteq \mathbb{R}^n$ will be an open set. Given the function $f$: $X \to \mathbb{R}$, the epigraph of $f$ is

$$\operatorname{epi} f := \{(x, y) \in X \times \mathbb{R} : f(x) \leq y\}.$$

The set $\operatorname{epi} f$ will be locally approximated at the point $(x, f(x))$ by a local cone approximation $K$ and a positively homogeneous function $f^K(x, \cdot)$ will be uniquely determined.

DEFINITION 1.1. Let $f$: $X \to \mathbb{R}$, $x \in X$ and $K$ be a local cone approximation; the positively homogeneous function $f^K(x, \cdot)$: $\mathbb{R}^n \to [-\infty, +\infty]$ defined by

$$f^K(x, y) := \inf\{\beta \in \mathbb{R} : (y, \beta) \in K(\operatorname{epi} f, (x, f(x)))\}$$

is called the *K-directional derivative* of $f$ at $x$.

By means of Definition 1.1 we can recover most of the generalized directional derivatives used in literature; for instance

- the *upper Dini directional derivative* of $f$ at $x$

$$D_+ f(x, y) := \limsup_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}$$

is associated to the *cone of the feasible directions*

$$F(Q,x) := \{y \in \mathbb{R}^n\colon \forall\{t_k\} \to 0^+,\ x + t_k y \in Q\};$$

- the *lower Dini directional derivative* of $f$ at $x$

$$D_-f(x,y) := \liminf_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}$$

is associated to the *cone of the weak feasible directions*

$$WF(Q,x) := \{y \in \mathbb{R}^n : \exists\{t_k\} \to 0^+ \text{ s.t. } x + t_k y \in Q\};$$

- if $f$ is locally Lipschitz, the *Clarke directional derivative* of $f$ at $x$

$$f^\circ(x,y) := \limsup_{(x',t) \to (x,0^+)} \frac{f(x'+ty) - f(x')}{t}$$

is associated to *Clarke's tangent cone*

$$T_{Cl}(Q,x) := \{y \in \mathbb{R}^n\colon \forall\{x_k\} \to x \text{ s.t. } x_k \in Q,$$
$$\forall\{t_k\} \to 0^+,\ \exists\{y_k\} \to y \text{ s.t. } x_k + t_k y_k \in Q\}.$$

It is well known that $f^\circ(x,\cdot) \geq D_+f(x,\cdot) \geq D_-f(x,\cdot)$. For a more detailed review about the local cone approximations we refer to [10].

DEFINITION 1.2. Let $f\colon X \to \mathbb{R}$, $x \in X$ and $K$ be a local cone approximation;

- $f$ is said to be *K-subdifferentiable* at $x$ if there exists a convex compact set $\partial^K f(x)$ such that

$$f^K(x,y) = \max_{x^* \in \partial^K f(x)} \langle x^*, y\rangle, \qquad \forall y \in \mathbb{R}^n;$$

the set $\partial^K f(x)$ is called the *K-subdifferential* of $f$ at $x$.

- $f$ is said *K-quasidifferentiable* at $x$ if there exist two convex compact sets $\underline{\partial}^K f(x)$ and $\overline{\partial}^K f(x)$ such that

$$f^K(x,y) = \max_{\underline{x}^* \in \underline{\partial}^K f(x)} \langle \underline{x}^*, y\rangle - \max_{\bar{x}^* \in \overline{\partial}^K f(x)} \langle \bar{x}^*, y\rangle, \qquad \forall y \in \mathbb{R}^n;$$

the sets $\underline{\partial}^K f(x)$ and $\overline{\partial}^K f(x)$ are called the *K-subdifferential* and *K-superdifferential* of $f$ at $x$, respectively.

*Remark* 1.1. It is immediate to observe that every $K$-subdifferentiable function is $K$-quasidifferentiable with $\overline{\partial}^K f(x) = \{0\}$ and $\underline{\partial}^K f(x) = \partial^K f(x)$.

It is well known that the Clarke derivative is bounded and convex; therefore there exists the Clarke subdifferential $\partial^\circ f := \partial^{T_{Cl}} f$. If $f$ is directionally differentiable (i.e., $D_- f(x, \cdot) = D_+ f(x, \cdot) := f'(x, \cdot)$) and it is $F$-subdifferentiable we say that $f$ is quasidifferentiable in the sense of Pshenichnyi [21], while if $f$ is $F$-quasidifferentiable we say that $f$ is quasidifferentiable in the sense of Demyanov and Rubinov [8].

DEFINITION 1.3. Let $f\colon X \to \mathbb{R}$ and $K$ be a local cone approximation; $x \in X$ is said to be a *$K$-inf-stationary point* for $f$ if $f^K(x, y) \geq 0$ for each $y \in \mathbb{R}^n$.

The following result gives the characterization of a $K$-inf-stationary point for $K$-quasidifferentiable functions.

THEOREM 1.1. *Let $f\colon X \to \mathbb{R}$ and $K$ be a local cone approximation. If $f$ is $K$-quasidifferentiable, then $x \in X$ is a $K$-inf-stationary point for $f$ if and only if $\overline{\partial}^K f(x) \subseteq \underline{\partial}^K f(x)$.*

*Proof.* Let $x$ be a $K$-inf-stationary point and let us suppose by contradiction that there exists $\bar{x}^* \in \overline{\partial}^K f(x)$ such that $\bar{x}^* \notin \underline{\partial}^K f(x)$. Then there exist $y \in \mathbb{R}^n$ and $\varepsilon > 0$ such that

$$\langle \bar{x}^*, y \rangle \geq \langle \underline{x}^*, y \rangle + \varepsilon, \qquad \forall \underline{x}^* \in \underline{\partial}^K f(x);$$

hence

$$\langle \bar{x}^*, y \rangle \geq \max_{\underline{x}^* \in \underline{\partial}^K f(x)} \langle \underline{x}^*, y \rangle + \varepsilon,$$

and then

$$-\varepsilon \geq \max_{\underline{x}^* \in \underline{\partial}^K f(x)} \langle \underline{x}^*, y \rangle - \langle \bar{x}^*, y \rangle \geq f^K(x, y)$$

that contradicts the assumption. The converse implication is immediate. ∎

*Remark* 1.2. In particular, if $f$ is $K$-subdifferentiable, then $x \in X$ is a $K$-inf-stationary point for $f$ if and only if $0 \in \partial^K f(x)$.

In [3, 4] it was proved that if $K$ is an isotone local approximation (i.e., $K(A, x) \subseteq K(B, x)$ for each $A \subseteq B$), then every local minimizer of $f$ over $\mathbb{R}^n$ is a $K$-inf-stationary point for $f$. Unfortunately, in general it is not possible to deduce that a $K$-inf-stationary point is a local optimal solution. For this reason, in Section 2, we will introduce the concept of $K$-invexity.

## 2. $K$-INVEXITY

In this section we propose a unifying definition of invexity for nonsmooth functions.

DEFINITION 2.1. Let $K$ be a local cone approximation; the function $f$: $X \to \mathbb{R}$ is said to be *$K$-invex* if there exists a function $\eta$: $X \times X \to \mathbb{R}^n$ such that

$$f(x_2) - f(x_1) \geq f^K(x_1, \eta(x_1, x_2)), \qquad \forall x_1, x_2 \in X.$$

The function $\eta$ is said to be the *kernel* of the $K$-invexity.

By means of Definition 2.1, we can obtain all the definitions of invexity for nonsmooth functions. For instance if we use Clarke's tangent cone for locally Lipschitz functions, we recover the concept of invexity introduced by Reiland [22]; if we consider the class of the directionally differentiable functions and we take $K = F$, we get the $d$-invexity given by Ye [26]. Moreover if $f$ is also $F$-subdifferentiable or $F$-quasidifferentiable, we obtain the $P$-invexity and the $DR$-invexity, respectively, studied in [12].

Finally, we observe that if $f^{K_1}(x, \cdot\,) \geq f^{K_2}(x, \cdot\,)$ and $f$ is $K_1$-invex then $f$ is $K_2$-invex with respect to the same kernel; in particular every locally Lipschitz $T_{Cl}$-invex function [22] is also $K$-invex for $K = F, WF$.

The following result shows the characterization of $K$-invexity for $K$-sub- (or quasi-) differentiable functions.

THEOREM 2.1. *Let $f$: $X \to \mathbb{R}$ and $K$ be a local cone approximation. If $f$ is $K$-quasidifferentiable, then $f$ is $K$-invex with respect to the kernel $\eta$ if and only if for each $x_1, x_2 \in X$ and for each $\underline{x}^* \in \underline{\partial}^K f(x_1)$ there exists $\bar{x}^*(x_1, x_2) \in \bar{\partial}^K f(x_1)$ such that*

$$f(x_2) - f(x_1) \geq \langle \underline{x}^* - \bar{x}^*(x_1, x_2), \eta(x_1, x_2) \rangle.$$

*Proof.* Let $f$ be $K$-invex; then for each $x_1, x_2 \in X$ and for each $\underline{x}^* \in \underline{\partial}^K f(x_1)$

$$\begin{aligned} f(x_2) - f(x_1) &\geq f^K(x_1, \eta(x_1, x_2)) \\ &\geq \langle \underline{x}^*, \eta(x_1, x_2) \rangle - \max_{\bar{x}^* \in \bar{\partial}^K f(x_1)} \langle \bar{x}^*, \eta(x_1, x_2) \rangle. \end{aligned}$$

Since $\bar{\partial}^K f(x_1)$ is a compact set, there exists $\bar{x}^*(x_1, x_2) \in \bar{\partial}^K f(x_1)$ such that

$$\max_{\bar{x}^* \in \bar{\partial}^K f(x_1)} \langle \bar{x}^*, \eta(x_1, x_2) \rangle = \langle \bar{x}^*(x_1, x_2), \eta(x_1, x_2) \rangle$$

and the thesis is achieved. For the converse, by assumption

$$\begin{aligned} f(x_2) - f(x_1) &\geq \max_{\underline{x}^* \in \underline{\partial}^K f(x_1)} \langle \underline{x}^*, \eta(x_1, x_2) \rangle - \langle \bar{x}^*(x_1, x_2), \eta(x_1, x_2) \rangle \\ &\geq \max_{\underline{x}^* \in \underline{\partial}^K f(x_1)} \langle \underline{x}^*, \eta(x_1, x_2) \rangle - \max_{\bar{x}^* \in \bar{\partial}^K f(x_1)} \langle \bar{x}^*, \eta(x_1, x_2) \rangle \\ &= f^K(x_1, \eta(x_1, x_2)). \end{aligned}$$

∎

*Remark* 2.1. In particular, if $f$ is $K$-subdifferentiable, then $f$ is $K$-invex with respect to the kernel $\eta$ if and only if for each $x_1, x_2 \in X$

$$f(x_2) - f(x_1) \geq \langle x^*, \eta(x_1, x_2) \rangle, \qquad \forall x^* \in \partial^K f(x_1).$$

To deepen the analysis of the structure of this class of functions, we consider a generalization of the result given by Craven and Glover [7] and Ben-Israel and Mond [1] and proved for differentiable functions and adapted by Giorgi and Guerraggio [12] for Lipschitz nonsmooth functions.

THEOREM 2.2. *Let $f$: $X \to \mathbb{R}$ and $K$ be a local cone approximation*; *$f$ is $K$-invex if and only if every $K$-inf-stationary point is a global minimum point.*

*Proof.* Consider the following two cases.

(a) If $x_1, x_2 \in X$ are such that $f(x_2) \geq f(x_1)$ we take $\eta(x_1, x_2) := 0$.

(b) If $x_1, x_2 \in X$ are such that $f(x_2) < f(x_1)$, then $x_1$ cannot be a $K$-inf-stationary point and therefore there exists a direction $y \in \mathbb{R}^n$ such that $f^K(x_1, y) < 0$. If we consider the function

$$\eta(x_1, x_2) := \frac{f(x_2) - f(x_1)}{f^K(x_1, y)} y$$

then

$$f^K(x_1, \eta(x_1, x_2)) = \frac{f(x_2) - f(x_1)}{f^K(x_1, y)} f^K(x_1, y) = f(x_2) - f(x_1),$$

hence $f$ is $K$-invex with respect to $\eta$. The converse implication is immediate. ∎

EXAMPLE 2.1. Given $f(x) = -\|x\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$, then

$$f^\circ(x, y) = \begin{cases} -\dfrac{1}{\|x\|}\langle x, y\rangle, & \text{if } x \neq 0, \\ \|y\|, & \text{if } x = 0, \end{cases}$$

$$f'(x, y) = \begin{cases} -\dfrac{1}{\|x\|}\langle x, y\rangle, & \text{if } x \neq 0, \\ -\|y\|, & \text{if } x = 0. \end{cases}$$

Since $\bar{x} = 0$ is a $T_{Cl}$-inf-stationary point but it is not a global minimum, for Theorem 2.2, $f$ is not $T_{Cl}$-invex; on the contrary $f$ is $F(WF)$-invex with respect to the kernel

$$\eta(x_1, x_2) = \begin{cases} \dfrac{\|x_2\|}{\|x_1\|}x_1, & \text{if } x_1 \neq 0, \\ x_2, & \text{if } x_1 = 0. \end{cases}$$

The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} |x|, & \text{if } x \in \mathbb{Q}, \\ -|x|, & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not $F$-invex because $\bar{x} = 0$ is a $F$-inf-stationary point $(D_+f(0, y) = |y|)$ but it is not a global minimum; on the contrary it is $WF$-invex with respect to the kernel

$$\eta(x_1, x_2) = \begin{cases} \dfrac{x_1}{|x_1|}(|x_1| - |x_2|), & \text{if } x_1 \neq 0, \\ |x_2|, & \text{if } x_1 = 0. \end{cases}$$

Straightforward extensions of $K$-invexity can be made as follows.

DEFINITION 2.2. Let $K$ be a local cone approximation; the function $f\colon X \to \mathbb{R}$ is said to be

- *$K$-pseudoinvex* if there exists a function $\eta\colon X \times X \to \mathbb{R}^n$ such that

$$f^K(x_1, \eta(x_1, x_2)) \geq 0 \Rightarrow f(x_2) \geq f(x_1), \qquad \forall x_1, x_2 \in X;$$

- *$K$-quasi-invex* if there exists a function $\eta\colon X \times X \to \mathbb{R}^n$ such that

$$f(x_2) \leq f(x_1) \Rightarrow f^K(x_1, \eta(x_1, x_2)) \leq 0, \qquad \forall x_1, x_2 \in X;$$

- *strictly K-quasi-invex* if there exists a functional $\eta$: $X \times X \to \mathbb{R}^n$ such that

$$f(x_2) \leq f(x_1) \Rightarrow f^K(x_1, \eta(x_1, x_2)) < 0, \qquad \forall x_1 \neq x_2 \in X.$$

We observe that every $K$-invex function is both $K$-pseudoinvex and $K$-quasi-invex with respect to the same kernel. In [19] it was shown the existence of a $K$-pseudoinvex function with respect to the kernel $\eta$ which is not $K$-invex with respect to the same $\eta$. Nevertheless, from the characterization given in Theorem 2.2 for $K$-invex functions, it is immediate to deduce that the two definitions coincide. In other words a $K$-pseudoinvex function may be not $K$-invex for the same $\eta$ but will be $K$-invex for some $\eta$. With the next proof we emphasize this fact and we show how the kernel of invexity can change.

THEOREM 2.3. *Let $f$: $X \to \mathbb{R}$ and $K$ be a local cone approximation; if $f$ is $K$-pseudoinvex then $f$ is $K$-invex.*

*Proof.* Let $f$ be $K$-pseudoinvex with respect to $\eta$ and, in order to show that $f$ is $K$-invex, we consider the following two cases.

(a) If $x_1, x_2 \in X$ are such that $f(x_2) \geq f(x_1)$ it is sufficient to take $\overline{\eta}(x_1, x_2) := 0$.

(b) If $x_1, x_2 \in X$ are such that $f(x_2) < f(x_1)$, then $f^K(x_1, \eta(x_1, x_2)) < 0$. If we consider the function

$$\overline{\eta}(x_1, x_2) := \frac{f(x_2) - f(x_1)}{f^K(x_1, \eta(x_1, x_2))} \eta(x_1, x_2)$$

then

$$\begin{aligned} f^K(x_1, \overline{\eta}(x_1, x_2)) &= \frac{f(x_2) - f(x_1)}{f^K(x_1, \eta(x_1, x_2))} f^K(x_1, \eta(x_1, x_2)) \\ &= f(x_2) - f(x_1), \end{aligned}$$

hence $f$ is $K$-invex with respect to $\overline{\eta}$. ∎

*Remark* 2.2. We observe that no assumption is required for proving Theorem 2.3. The same result was established by Giorgi and Guerraggio [12] for the case of locally Lipschitz functions and via Clarke's tangent cone. Moreover, the same result was obtained by Reiland [22] but under the unnecessary condition that the cone

$$\bigcup_{\lambda \geq 0} \left( \lambda \partial^\circ f(x_1) \times \{\lambda(f(x_2) - f(x_1))\} \right)$$

be closed.

## 3. SUFFICIENT OPTIMALITY CONDITIONS

In this section we study the extremum problem

$$\min\{f_0(x): f_i(x) \le 0,\ i \in I\}, \tag{$P$}$$

where $X \subseteq \mathbb{R}^n$ is an open set, $f_0, f_i$: $X \to \mathbb{R}$, and $I := \{1, \dots, m\}$. For every feasible point $x$ we denote $I(x) := \{i \in I: f_i(x) = 0\}$; moreover $I_0 := I \cup \{0\}$ and $I_0(x) := I(x) \cup \{0\}$.

In the last two decades many generalizations of the Kuhn–Tucker necessary optimality condition for the problem $(P)$ have been stated without assuming the differentiability of the functions $f_i$ (see [5, 16, 23, 24] and references therein). It has been proved (see for instance [3, 4]) that these necessary optimality conditions are equivalent to the impossibility of suitable systems of sublinear functions.

Definition 3.1. Let $\bar{x}$ be a feasible point for $(P)$ and $K_i$, with $i \in I_0(\bar{x})$, be local cone approximations; the point $\bar{x}$ is said to be:

- a *weakly stationary point* for the problem $(P)$ with respect to $K_i$ if the following system is impossible

$$\begin{cases} f_0^{K_0}(\bar{x}, y) < 0, \\ f_i^{K_i}(\bar{x}, y) < 0, \qquad i \in I(\bar{x}); \end{cases} \tag{$S_1$}$$

- a *strongly stationary point* for the problem $(P)$ with respect to $K_i$ if the following system is impossible

$$\begin{cases} f^{K_0}(\bar{x}, y) < 0, \\ f_i^{K_i}(\bar{x}, y) \le 0, \qquad i \in I(\bar{x}). \end{cases} \tag{$S_2$}$$

In [3, 4, 23, 25] it was shown that it is always possible to choose suitable local cone approximations $K_i$, with $i \in I_0(\bar{x})$, which do not depend on the functions $f_i$ and such that every local optimal solution $\bar{x}$ is a weakly stationary point with respect to $K_i$. Such local cone approximations are called *admissible*. For instance, $K_0 = WF$ and $K_i = F$, or, if $f_i$ are locally Lipschitz functions, $K_0 = K_i = T$, are admissible.

Moreover, if some regularity condition holds, it is possible to prove [2] that every weakly stationary point is a strongly stationary point.

Finally, if $f_i^{K_i}(\bar{x}, \cdot\,)$, with $i \in I_0(\bar{x})$, are convex (or difference of convex functions or, more generally, pointwise minimum of sublinear functions), it is possible to prove, through theorems of the alternative [13, 14], that the impossibility of the systems $(S_1)$ and $(S_2)$ are equivalent to the generalizations of the John and Kuhn–Tucker necessary optimality conditions, respectively. In other words if, for instance, $f_i^{K_i}(\bar{x}, \cdot)$, with $i \in I_0(\bar{x})$, are convex, the impossibility of $(S_1)$ is equivalent to the existence of the John multipliers $\lambda_i \geq 0$, with $i \in I_0(\bar{x})$, not all zero, such that

$$0 \in \sum_{i \in I_0(\bar{x})} \lambda_i \partial^{K_i} f_i(\bar{x}); \tag{$J$}$$

while the impossibility of $(S_2)$, under a suitable regularity condition, is equivalent to the existence of the Kuhn–Tucker multipliers $\lambda_i \geq 0$, with $i \in I(\bar{x})$, such that

$$0 \in \partial^{K_0} f_0(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial^{K_i} f_i(\bar{x}). \tag{$KT$}$$

All the sufficiency results stated for nonsmooth invex functions are deduced from the necessary optimality conditions $(J)$ or $(KT)$; hence they require the convexity of the directional derivatives [19, 22, 26]. The results of this section show that, under suitable assumptions of invexity, it is possible to deduce sufficient optimality conditions directly from the impossibility of the systems $(S_1)$ and $(S_2)$.

THEOREM 3.1. *Let $\bar{x}$ be a strongly stationary point for the problem $(P)$ with respect to $K_i$, $i \in I_0(\bar{x})$. If $f_0$ is $K_0$-pseudoinvex and $f_i$ are $K_i$-quasi-invex with respect to the same kernel $\eta$ then $\bar{x}$ is a global optimal solution for $(P)$.*

*Proof.* Let $x$ be any feasible point for $(P)$. Then

$$f_i(x) \leq 0 = f_i(\bar{x}), \qquad \forall i \in I(\bar{x}).$$

By the $K_i$-quasiinvexity of $f_i$, we have

$$f_i^{K_i}(\bar{x}, \eta(\bar{x}, x)) \leq 0, \qquad \forall i \in I(\bar{x}).$$

Since $(S_2)$ is impossible

$$f_0^{K_0}(\bar{x}, \eta(\bar{x}, x)) \geq 0,$$

and, by the $K_0$-pseudoinvexity of $f_0$, we conclude $f_0(x) \geq f_0(\bar{x})$. ∎

EXAMPLE 3.1. Given the problem

$$\min f_0(x_1, x_2) = x_1^2 - 2|x_1| + |x_2|,$$
$$f_1(x_1, x_2) = x_1^4 + 5|x_1| - 2|x_2| \leq 0,$$

if we consider the admissible pair $(WF, F)$, we have

$$f_0'((x_1, x_2), (\eta_1, \eta_2)) = 2x_1\eta_1 - 2\varphi(x_1, \eta_1) + \varphi(x_2, \eta_2),$$
$$f_1'((x_1, x_2), (\eta_1, \eta_2)) = 4x_1^3\eta_1 + 5\varphi(x_1, \eta_1) - 2\varphi(x_2, \eta_2),$$

where $\varphi: \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\varphi(x, \eta) = \begin{cases} \eta, & \text{if } x > 0, \\ |\eta|, & \text{if } x = 0, \\ -\eta, & \text{if } x < 0. \end{cases}$$

It is immediate to see that $\bar{x} = (0, 0)$ is a strongly stationary point, that is, the following system is impossible

$$\begin{cases} f_0'((0,0), (\eta_1, \eta_2)) = -2|\eta_1| + |\eta_2| < 0, \\ f_1'((0,0), (\eta_1, \eta_2)) = 5|\eta_1| - 2|\eta_2| \leq 0. \end{cases}$$

Since $f_0$ is $WF$-pseudoinvex and $f_1$ is $F$-quasi-invex with respect to the same kernel

$$\eta((x_1, x_2), (y_1, y_2)) = \big(-x_1 + \varphi(x_1, |y_1|),\ -x_2 + \varphi(x_2, |y_2|)\big),$$

the conditions of Theorem 3.1 are satisfied and $\bar{x} = (0, 0)$ is an optimal solution of $(P)$.

Some remarks are needed: first of all we observe that $f_i'((0,0), (\eta_1, \eta_2))$ are not sublinear and therefore we cannot consider a necessary optimality condition of Kuhn–Tucker-type with subdifferentials [26]. Nevertheless the functions are quasidifferentiable [8] and therefore it is possible to have a necessary optimality condition expressed via quasidifferentials [8]. Moreover, since $f_i$ are Lipschitzian, we can exploit the Clarke's directional derivative. In such a way we get the impossibility of the system

$$\begin{cases} f_0^\circ((0,0), (\eta_1, \eta_2)) = 2|\eta_1| + |\eta_2| < 0, \\ f_0^\circ((0,0), (\eta_1, \eta_2)) = 5|\eta_1| + 2|\eta_2| \leq 0. \end{cases}$$

We observe that $\bar{x} = (0, 0)$ is a $T_{Cl}$-inf-stationary point for $f_0$ but it is not a global minimum; therefore $f_0$ is not $T_{Cl}$-pseudoinvex. Hence we cannot apply Theorem 3.1 and related theory [19, 22].

We have noted that the impossibility of $(S_2)$ descends from the impossibility of $(S_1)$ in the presence of a regularity condition. Nevertheless, even if we do not have regularity but we strengthen the hypothesis of invexity of the constraint functions, the impossibility of the system $(S_1)$ implies the optimality of $\bar{x}$.

THEOREM 3.2. *Let $\bar{x}$ be a weakly stationary point for the problem $(P)$ with respect to $K_i$, $i \in I_0(\bar{x})$. If $f_0$ is $K_0$-pseudoinvex and the $f_i$ are strictly $K_i$-quasi-invex with respect to the same kernel $\eta$ then $\bar{x}$ is a global optimal solution for $(P)$.*

*Proof.* The proof is the same as that of Theorem 3.1 except that, by the strict $K_i$-quasi-invexity of $f_i$ we can deduce $f_i^{K_i}(\bar{x}, \eta(\bar{x}, x)) < 0$. ∎

## 4. DUALITY

We conclude giving weak and strong duality results for problems with $K_i$-subdifferentiable functions. Consider the following modified Mond–Weir [20] dual problem:

$$\begin{cases} \max f(u) \\ 0 \in \partial^{K_0} f_0(u) + \sum_{i \in I} \lambda_i \partial^{K_i} f_i(u), \\ \lambda_i f_i(u) \geq 0, \\ \lambda_i \geq 0. \end{cases} \tag{$D$}$$

The following duality results are established for $(P)$ and $(D)$.

THEOREM 4.1 (Weak Duality). *Let $x$ be feasible for $(P)$ and $(u, \lambda)$ be feasible for $(D)$. If $f_0$ is $K_0$-pseudoinvex and $f_i$ is $K_i$-quasi-invex with respect to the same kernel $\eta$, then $f_0(x) \geq f_0(u)$.*

*Proof.* Since $\lambda_i \geq 0$ and $f_i(x) \leq 0$ we have $\lambda_i f_i(x) \leq \lambda_i f_i(u)$. By $K$-quasi-invexity of $f_i$ we obtain

$$\langle u_i^*, \eta(x, u) \rangle \leq 0, \qquad \forall u_i^* \in \partial^{K_i} f_i(u).$$

Moreover, by feasibility of $(u, \lambda)$, there exists $u_i^* \in \partial^{K_i} f_i(u)$ such that

$$u_0^* + \sum_{i \in I} \lambda_i u_i^* = 0.$$

Therefore

$$0 \geq \sum_{i \in I} \lambda_i \langle u_i^*, \eta(x, u) \rangle = \langle -u_0^*, \eta(x, u) \rangle.$$

Since $f_0$ is $K$-pseudoinvex we achieve the thesis. ∎

THEOREM 4.2 (Strong Duality). *Let $\bar{x}$ be a local minimum of $(P)$ and assume that a regularity condition holds at $\bar{x}$. Then there exists $\bar{\lambda}$ such that*

*$(\bar{x}, \bar{\lambda})$ is feasible for $(D)$. Moreover if $f_0$ is $K_0$-pseudoinvex and $f_i$ is $K_i$-quasi-invex with respect to the same kernel $\eta$, then $\bar{x}$ and $(\bar{x}, \bar{\lambda})$ are global minima of $(P)$ and $(D)$, respectively.*

*Proof.* By assumption there exists $\bar{\lambda}$ such that $(\bar{x}, \bar{\lambda})$ is feasible for $(D)$. From Theorem 4.1 we achieve the thesis. ∎

When the $f_i$ are locally Lipschitz, choosing $K_i = T_{Cl}$, we get a result similar to one expressed in [19, 22].

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